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Evolution of inertial frequency oscillations

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Approximate equations are derived for the slow variation in amplitude of free oscillations at the inertial frequency in a slightly stratified ocean. In an ocean of constant depth with horizontally uniform stratification the evolution equation for waves of small vertical mode number can be reduced to a Schrödinger equation, with the increment in Coriolis parameter playing the role of the potential. An exact solution of the Schrödinger equation is presented which demonstrates the transience of inertial oscillations.

1. Introduction

An almost universal feature of horizontal current measurements in the ocean is a rotating current with frequency close to the inertial frequency f, and a speed of several centimetres per second. The vertical and lateral scales of these oscillations range from 10 m and 10 km respectively (Webster 1968) to 200 m and 1000 km (R. T. Pollard, private communication). In view of the strength and widespread occurrence of inertial oscillations, one might expect that their horizontal and vertical structure could readily be determined from current measurements. Unfortunately, this is not the case, because the amplitude and structure are found to vary markedly on a time scale as short as ten periods (Webster 1968). Moreover, a change in the wind strength or direction produces new waves at the inertial frequency which tend to mask the original waves.

Transience would then seem to be a major characteristic of inertial oscillations. Estimates of the turbulent decay rate for internal waves of the inertial frequency give time scales of the order of 100 inertial periods (LeBlond 1966). Thus the transience of the oscillations is due to more subtle mechanisms. Munk & Phillips (1968) examine the solution for periodic inertio-gravity waves in a shallow ocean of constant depth on a rotating sphere. The wave energy is essentially confined between two latitudes, either side of the equator, for which the waves are of inertial frequency. Close to these critical latitudes, not only is there a caustic of the ray paths, but also the group velocity is small and the wave amplitudes are at their largest. Thus Munk & Phillips suggest that inertial oscillations are gravity waves at a 'turning latitude' and that the transience is due to the wave energy being slowly propagated towards the equator. However, they show that neither of the two extreme models of homogeneous random sources and local coherent generation can separately account for the observed rate of transience. Another mechanism which can explain transience even in small seas is the phase mixing of a large number of wave modes (Hasselmann 1970). For the Baltic and Mediterranean, respectively, Hasselmann (1970) and

Perkins (1972) have shown that this explanation yields decay rates in reasonable agreement with observations. The related effect of wavelength dispersion was considered by Pollard (1970), but he found that the predicted decay rate was too small to explain transience in an open ocean. We speculate that wavelength dispersion may be the dominant mechanism for a long narrow sea of small north-to-south extent (see §3).

In the present paper, we are concerned with the mathematical question of determining an equation, incorporating all the above-mentioned mechanisms, which governs the slow evolution of the amplitude of inertial oscillations. The major mathematical tool that is used is the method of multiple time scales (Cole 1968), a short time scale describing the rotation of the horizontal current and a long time scale describing the transience. In organizing the calculations we are guided by the observed properties of the oceans and of inertial oscillations. For example, use is made of the fact that the Brunt–Väisälä (or buoyancy) frequency N is large relative to f, and also that the vertical scale of the waves is considerably less than their horizontal scale.

For the idealized case of waves with small vertical mode number in an ocean which has horizontally uniform stratification, constant depth and a barotropic mean current the evolution equation can be reduced to a Schrödinger equation, with the increment in f plus *half* the current vorticity playing the role of the potential. Thus, the leakage of wave energy towards the equator is equivalent to reflexion at a potential barrier, and a strong current can act as a potential well.

2. Equations and boundary conditions

The linearized Boussinesq equations of motion for a stratified inviscid fluid are given in Phillips (1966, chap. 2). These, together with the boundary conditions for a free surface at $z = 0$ and a rigid 'bottom topography' $z = -D(x)$, can be combined into the form

$$\frac{\partial^3}{\partial z^2\,\partial t}\left[\frac{\partial \mathbf{q}}{\partial t}+f_0\mathbf{k}\times\mathbf{q}\right]+\frac{\partial^3}{\partial z^2\,\partial t}[(f-f_0)\,\mathbf{k}\times\mathbf{q}]+\nabla_h[N^2\nabla_h.\mathbf{q}]$$
$$+2\frac{\partial^2}{\partial z\,\partial t}[\nabla_h(\mathbf{q}.(\mathbf{\Omega}\times\mathbf{k}))-(\mathbf{\Omega}\times\mathbf{k})\,\nabla_h.\mathbf{q}]+\nabla_h\left[\frac{\partial^2}{\partial t^2}\nabla_h.\mathbf{q}\right]=0, \qquad (1a)$$

$$\frac{\partial^2}{\partial z\,\partial t}\left[\frac{\partial \mathbf{q}}{\partial t}+f_0\mathbf{k}\times\mathbf{q}\right]+\frac{\partial^2}{\partial z\,\partial t}[(f-f_0)\,\mathbf{k}\times\mathbf{q}]-g\nabla_h[\nabla_h.\mathbf{q}]$$
$$+2(\mathbf{\Omega}\times\mathbf{k})\frac{\partial}{\partial t}(\nabla_h.\mathbf{q})=0 \quad \text{on} \quad z=0 \qquad (1b)$$

and
$$q = 0 \quad \text{on} \quad z = -D(\mathbf{x}), \qquad (1c)$$

where f_0 is the value of the Coriolis parameter at the origin of the chosen coordinate system, $\mathbf{k}$ is the vertical unit vector, $\mathbf{x}$ the horizontal position vector, $N(\mathbf{x},z)$ the buoyancy frequency, $D(\mathbf{x})$ the water depth, $\mathbf{\Omega}$ the horizontal component of the earth's angular velocity vector, ∇_h the horizontal gradient operator, and $\mathbf{q}$ is the vertically integrated horizontal current, i.e.

$$\mathbf{q}=\int_{-D}^{z}\mathbf{u}\,dz.$$

The choice of $\mathbf{q}$ as dependent variable is particularly convenient if we wish to study the effect of horizontal variations in D, f or N. We note that $-\nabla_h \cdot \mathbf{q}$ is the vertical velocity component, and it is this fact, and not (1 c), that corresponds to the rigid-bottom boundary condition.

If we let L and H denote typical length and depth scales for waves of inertial frequency, then for mid-latitudes we can estimate the ratio of the groups of terms in (1 a) as

$$1:\frac{L}{R}:\frac{\mathscr{N}^2}{f_0^2}\frac{H^2}{L^2}:\frac{H}{L}:\frac{H^2}{L^2},$$

where R is the earth's radius and $\mathscr{N}$ is a representative value of the buoyancy frequency. (For waves which are concentrated near the thermocline, a natural choice for $\mathscr{N}$ would be the maximum value of $N(\mathbf{x}, z)$, however, for other classes of waves the mean value of N may be a more meaningful choice.) Typical values for scales of inertial frequency oscillations are

$$L = 100\,\text{km}, \quad H = 100\,\text{m}, \quad \mathscr{N}/f_0 = 50,$$

and the ratio of the groups of terms in (1 a) is then estimated as

$$1:1{\cdot}5\times 10^{-2}:2{\cdot}5\times 10^{-3}:10^{-3}:10^{-6}.$$

Thus in (1 a) the first group of terms is dominant and a first approximation for $\mathbf{q}$ yields the well-known result that the horizontal current rotates clockwise in the northern hemisphere:

$$\mathbf{u} = \partial\mathbf{q}/\partial z = \partial[\mathbf{a}\cos f_0 t - \mathbf{k}\times\mathbf{a}\sin f_0 t]/\partial z, \tag{2}$$

where $\mathbf{a}$ is an undetermined function of (x, z) and can vary slowly with time.

In order to determine the possible spatial dependence of $\mathbf{a}$ and its slow evolution in time, we must examine higher approximations to (1 a). A self-consistent theory for waves with small vertical mode number can be developed if the last two groups of terms in (1 a) are neglected, that is if the effects of $\mathbf{\Omega}$ and of vertical accelerations are both neglected. Since, if H greatly exceeds $R(f/\mathscr{N})^4$, typically 1 m, then for all values of L either the buoyancy term or the $f-f_0$ term will dominate the neglected terms. However, we shall first consider the most complicated possibility, in which the second, third and fourth groups of terms in (1 a) are of the same small order.

We define a small parameter ϵ by

$$\epsilon = f_0^2/\mathscr{N}^2$$

and we non-dimensionalize the independent variables t, z and $\mathbf{x}$ with respect to and f_0, $\epsilon^2 R$ and ϵR respectively (typically 1 day, 1 m and 2·5 km respectively). Likewise, we non-dimensionalize the physical parameters which occur in (1):

$$\gamma = (f-f_0)/\epsilon f_0, \quad \tilde{N} = N/\mathscr{N}, \quad \tilde{\mathbf{\Omega}} = \mathbf{\Omega}/f_0, \quad G = g/f_0^2 R.$$

In dimensionless form, equations (1) become

$$\frac{\partial^3}{\partial z^2\,\partial t}\left[\frac{\partial\mathbf{q}}{\partial t}+\mathbf{k}\times\mathbf{q}\right]+\epsilon\frac{\partial^3}{\partial z^2\,\partial t}[\gamma\mathbf{k}\times\mathbf{q}]+\epsilon\nabla_h[\tilde{N}^2\nabla_h\cdot\mathbf{q}]$$
$$+2\epsilon\frac{\partial^2}{\partial z\,\partial t}[\nabla_h(\mathbf{q}\cdot(\tilde{\mathbf{\Omega}}\times\mathbf{k}))-\tilde{\mathbf{\Omega}}\times\mathbf{k}(\nabla_h\cdot\mathbf{q})]+\epsilon^2\nabla_h\left[\frac{\partial^2}{\partial t^2}\nabla_h\cdot\mathbf{q}\right]=0, \tag{3a}$$

$$\frac{\partial^2}{\partial z\,\partial t}\left[\frac{\partial \mathbf{q}}{\partial t}+\mathbf{k}\times\mathbf{q}\right]-G\nabla_h[\nabla_h.\mathbf{q}]+\epsilon\frac{\partial^2}{\partial z\,\partial t}[\gamma\mathbf{k}\times\mathbf{q}]$$
$$+2\epsilon(\tilde{\mathbf{\Omega}}\times\mathbf{k})\frac{\partial}{\partial t}(\nabla_h.\mathbf{q})=0 \quad \text{on} \quad z=0 \qquad (3b)$$

and
$$\mathbf{q}=0 \quad \text{on} \quad z=-D(\mathbf{x}). \qquad (3c)$$

Next, we use an expansion (Cole 1968, chap. 3)

$$\mathbf{q}=\mathbf{q}_0(\mathbf{x},z,t,T)+\epsilon\mathbf{q}_1(\mathbf{x},z,t,T)+\ldots,$$

with two time variables t and T, which denotes the long time scale associated with the evolution of the inertial oscillations. The operator $\partial/\partial t$ is now replaced by $\partial/\partial t+\epsilon\,\partial/\partial T$. Extracting the lowest order terms from (3a), we deduce that $\mathbf{q}_0$ is given by (2), or in dimensionless form

$$\mathbf{q}_0=\mathbf{a}\cos t-\mathbf{k}\times\mathbf{a}\sin t.$$

From the leading power of ϵ in the boundary condition (3b), it now follows that both $\nabla_h.\mathbf{a}$ and $\nabla_h.(\mathbf{k}\times\mathbf{a})$ are constant on the free surface. At a rigid coastline the component of $\mathbf{q}_0$ normal to the shoreline must be zero, and owing to the cyclic behaviour of $\mathbf{q}_0$ we infer that $\mathbf{a}$ is zero on the shoreline. Using the divergence theorem we can deduce that, for a finite sea, $\mathbf{a}$ is identically zero on the free surface. For an infinite sea, we get the same result if we assume that the wave energy is finite.

From the order ϵ terms in (3a) we can derive the result

$$\partial^2\mathbf{q}_1/\partial t^2=(\mathbf{a}_1+\mathbf{A})\cos t+(\mathbf{a}_1-\mathbf{A})\times\mathbf{k}\sin t-\mathbf{B}t\cos t-\mathbf{B}\times\mathbf{k}\,t\sin t,$$

where $\mathbf{a}_1$ is undetermined,

$$4\mathbf{A}=\nabla_h\times(\tilde{N}^2\nabla_h\times\mathbf{a})+\nabla_h(\tilde{N}^2\nabla_h.\mathbf{a})$$
$$+2\partial\{\tilde{\mathbf{\Omega}}\times(\nabla_h\times\mathbf{a})-\tilde{\mathbf{\Omega}}(\nabla_h.\mathbf{a})+(\tilde{\mathbf{\Omega}}.\nabla_h)\mathbf{a}+\mathbf{k}\times[(\tilde{\mathbf{\Omega}}\times\mathbf{k}).\nabla_h\mathbf{a}]\}/\partial z$$

and

$$\mathbf{B}=\frac{\partial^2}{\partial z^2}\left(\frac{\partial\mathbf{a}}{\partial T}+\gamma\mathbf{k}\times\mathbf{a}\right)+\tfrac{1}{2}\mathbf{k}\times\{\nabla_h(\tilde{N}^2\nabla_h.\mathbf{a})-\nabla_h\times(\tilde{N}^2\nabla_h\times\mathbf{a})\}$$
$$+\partial\{((\tilde{\mathbf{\Omega}}\times\mathbf{k}).\nabla_h)\mathbf{a}+\mathbf{k}\times((\tilde{\mathbf{\Omega}}.\nabla_h)\mathbf{a})-(\tilde{\mathbf{\Omega}}\times\mathbf{k})(\nabla_h.\mathbf{a})-\tilde{\mathbf{\Omega}}(\mathbf{k}.(\nabla_h\times\mathbf{a}))\}/\partial z.$$

For simplicity, derivatives of $\tilde{\mathbf{\Omega}}$ have been neglected in these expressions.

If $\mathbf{q}_0$ is to be a uniformly valid approximation to $\mathbf{q}$ then it is necessary that $\epsilon\mathbf{q}_1$ remains small for all time. This is possible only if $\mathbf{B}$ is zero. Reverting to the dimensional co-ordinates, this requirement means that $\mathbf{a}$ must evolve according to the equation

$$\frac{\partial^2}{\partial z^2}\left(\frac{\partial\mathbf{a}}{\partial t}+(f-f_0)\,\mathbf{k}\times\mathbf{a}\right)+\frac{1}{2f_0}\mathbf{k}\times(\nabla_h(N^2\nabla_h.\mathbf{a})-\nabla_h\times(N^2\nabla_h\times\mathbf{a}))$$
$$+\partial\{[(\mathbf{\Omega}\times\mathbf{k}).\nabla_h]\mathbf{a}+\mathbf{k}\times[(\mathbf{\Omega}.\nabla_h)\mathbf{a}]-(\mathbf{\Omega}\times\mathbf{k})(\nabla_h.\mathbf{a})-\mathbf{\Omega}[\mathbf{k}.(\nabla_h\times\mathbf{a})]\}/\partial z=0. \qquad (4)$$

It can be shown that if we were to use any other scaling, then we would derive an equation of the same form as (4) except that certain terms would be omitted.

For example, if $H = \epsilon^{\frac{4}{3}}R$ and $L = \epsilon^{\frac{2}{3}}R$ (typically 200 m and 40 km) then both the $f-f_0$ and the $\mathbf{\Omega}$ terms would be omitted.

The above calculation procedure can readily be adapted to include additional physical effects. For the particular case of an ocean current, we can estimate *a priori* that, if the horizontal speed exceeds $f_0 H$ (at most 1 cm s^{-1}), then the $\mathbf{\Omega}$ terms in (4) are smaller than the ignored terms due to the horizontal current. The appropriate modification to (4) is the inclusion of the extra terms

$$\frac{\partial}{\partial z}\left\{(\mathbf{U}\,.\,\nabla_h)\frac{\partial \mathbf{a}}{\partial z} + W\frac{\partial^2 \mathbf{a}}{\partial z^2} - \tfrac{1}{2}(\nabla_h\,.\,\mathbf{a})\frac{\partial}{\partial z}\mathbf{U} + \tfrac{1}{2}(\nabla_h\,.\,\mathbf{U})\frac{\partial \mathbf{a}}{\partial z} + \tfrac{1}{2}[\mathbf{k}\,.\,(\nabla_h\times\mathbf{U})]\,\mathbf{k}\times\frac{\partial \mathbf{a}}{\partial z} - \tfrac{1}{2}[\mathbf{k}\,.\,(\nabla_h\times\mathbf{a})]\,\mathbf{k}\times\frac{\partial \mathbf{U}}{\partial z}\right\},$$

where $\mathbf{U}$ and W respectively are the horizontal and vertical components of the current velocity. We note that if the horizontal current greatly exceeds $\epsilon^2 R f_0$ then it is justifiable to neglect the $\mathbf{\Omega}$ terms, since, for all values of H and L, there will be at least one term in the extended version of (4) which will greatly dominate the $\mathbf{\Omega}$ terms.

3. Reduction to Schrödinger's equation

The evolution equation (4) is much too complex to be studied analytically. Thus to make further progress we must either resort to numerical calculations or make additional simplifications. Here, we choose the latter alternative and assume that $\mathbf{\Omega}$ can be ignored, $\mathbf{U}$ is depth independent, N^2 is a function of z only and that the ocean depth is constant. Under these restrictions, (4) with the zero boundary conditions can be solved by a vertical eigenfunction expansion:

$$\mathbf{a} = \sum_j (\operatorname{Re}\psi_j, \operatorname{Im}\psi_j, 0)\,\phi_j(z),$$

where the functions ϕ_j satisfy the eigenvalue problem

$$\lambda^2\frac{d^2\phi}{dz^2} + \frac{N^2}{f_0^2}\phi = 0,$$

$$\phi = 0 \quad \text{on} \quad z = 0, -D$$

and the complex scalars ψ_j evolve according to the Schrödinger equation

$$\frac{i}{f_0}\left(\frac{\partial}{\partial t} + \mathbf{U}\,.\,\nabla_h\right)\psi = -\frac{\lambda^2}{2}\nabla_h^2\psi + \frac{1}{f_0}\left(f - f_0 + \tfrac{1}{2}\mathbf{k}\,.\,(\nabla_h\times\mathbf{U})\right)\psi. \tag{5}$$

For the particular case of a β-plane ocean that is bounded by lines of latitude and has zero current, we can, in principle, get an analytic solution to (5). First, we represent f as $f_0 + \beta y$ and the initial data as

$$\psi = \sum_l b_l(x, 0)\,\xi_l(y),$$

where the horizontal eigenfunctions ξ_l with the corresponding eigenvalues μ satisfy

$$\frac{d^2\xi}{dy^2}+\frac{1}{2\lambda^2}\left(\mu-\frac{\beta}{f_0}y\right)\xi = 0,$$

$$\xi = 0 \quad \text{on} \quad y = \pm L.$$

The solution to (5) can then be represented as

$$\psi = \sum_l \xi_l(y)\exp(-i\mu_l f_0 t)\left[\frac{-i}{2\pi\lambda^2 f_0 t}\right]^{\frac{1}{2}}\int_{-\infty}^{\infty} b_l(x',0)\exp\left[\frac{i(x-x')^2}{2\lambda^2 f_0 t}\right]dx'.$$

If only a few horizontal modes are strongly excited and $\mathscr{L}$ is used to denote a typical length scale for the initial data, then we can estimate that it takes a time of order $(\mathscr{L}/\lambda)^2$ inertial periods for the waves to reach the final algebraic decay rate of $t^{-\frac{1}{2}}$. The damping is entirely due to wavelength dispersion which, in the quantum mechanics analogy, is a consequence of the uncertainty principle. However, if a large number of modes are excited, then it is more appropriate to regard the ocean as being of infinite north-to-south extent and to use what is essentially the solution of Munk & Phillips (1968):

$$\psi = \left[\frac{-i}{2\pi\lambda^2 f_0 t}\right]^{\frac{1}{2}}\int_{-\infty}^{\infty}\exp\left[\frac{i(x-x')^2}{2\lambda^2 f_0 t}\right]\int_{-\infty}^{\infty} b(x',0,\mu) \times \mathrm{Ai}\left[y\left(\frac{\beta}{2\lambda^2 f_0}\right)^{\frac{1}{3}}-\mu\left(\frac{f_0^2}{2\lambda^2\beta^2}\right)^{\frac{1}{3}}\right]\exp(-i\mu f_0 t)\,d\mu\,dx'.$$

The final decay rate is now $t^{-\frac{3}{4}}$ but we get the same estimate as before for the time taken to reach the asymptotic state. In the quantum mechanics analogy the extra asymptotic rate of decay is due to the reflexion of waves away from regions of high potential, i.e. where $f-f_0$ is large.

For the oceans we can estimate that for the lowest mode λ would typically be 20 km and for the waves to have the observed transience scales of only ten inertial periods it is necessary that the initial data have a typical length scale of only 60 km. For waves in an open ocean caused by atmospheric disturbances such a length scale is rather short and we are led to the same conclusion as was made by Pollard (1970). Namely, that it is probable that wind stress must be invoked to destroy as well as create inertial oscillations.

4. Conclusion

The primary object of this paper was to derive the equations (4) which govern the slow evolution of the amplitude of inertial frequency oscillations, using scaling assumptions appropriate to the oceans. This evolution equation includes the effects of the horizontal components of the earth's rotation vector and it is possible to determine circumstances under which it is justifiable to make the 'traditional approximation' and neglect these terms. For example, if the horizontal current speed greatly exceeds $Rf_0^5/\mathscr{N}^4$, typically 10^{-2} cm s^{-1}, then, for waves of any horizontal and vertical scales, the traditional approximation is justifiable. A major characteristic of inertial oscillations in the oceans is their transience and a time-dependent solution to an idealized problem is presented which demonstrates that (4) can qualitatively describe transience.

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